Int. J. Solids Structures, 1970, Vol. 6, pp. 677 to 691. Pergamon Press. Printed in Great Britain

THE ELASTIC STABILITY OF STRUCTURAL SYSTEMS WITH INDEPENDENT LOADING PARAMETERS

KONCAY HUSEYIN†

Department of Civil Engineering, Middle East Technical University, Ankara, Turkey

Abstract—The stability of a general elastic structural system described by N generalized coordinates and M loading parameters is studied. Attention is restricted to conservative systems whose loss of stability is associated with a "general" critical point. Explicit asymptotic relationships describing the equilibrium surface in the vicinity of the critical point are derived intrinsically. The introduction of a certain transformation allows the local properties of the equilibrium surface and the nature of buckling to be examined conveniently. A theorem concerning the stability distribution on the equilibrium surface is proved.

Particular emphasis is given to the stability boundary of the system. It is found that a close relationship between the type of the critical point and the shape of the stability boundary exists. Three theorems concerning the basic properties of the stability boundary are established.

I. INTRODUCTION

IT HAS been shown [1, 2] that the buckling behaviour of structures under combined loading can normally be described either by a "general" or a "special" critical point on the equilibrium surface. The former is generally associated with a limit point, but under some conditions bifurcation of solution can also be obtained at the same critical point. The latter, on the other hand, is a genuine bifurcation point at which a simple extremum is definitely ruled out.

In the literature there exist a substantial number of investigations concerned with the latter case. In fact, the loss of stability of frames, plates, cylindrical and conical shells subjected to certain combinations of axial compression, shear, external pressure, etc., is often associated with a "special" critical point. In contrast, the existing work which can be associated with a "general" critical point is very limited. The experimental [3] and theoretical [4] investigations of Evan-Ivanowski and Loo on the buckling behaviour of a shallow spherical shell under the action of combined uniform pressure and concentrated load at the apex form an example of the "general" case. As another example Sabir's work [5] on deep spherical shells can be mentioned.

In the development of the general theory of elastic stability it is usually assumed [6–9] that the external loading of structures can be described by a single variable parameter. Sewell [10] in presenting the static perturbation technique for stability problems has started his analysis with a potential energy function comprising several parameters, but due to the introduction of a single perturbation parameter at an early stage of the analysis, the complete explicit equations of the equilibrium surface were not obtained. Furthermore the stability boundary of the system was not examined.

The determination of a stability boundary arises as a significant problem in connection with combined loading. Papkovich [11] considering the bifurcation buckling of a system

† Now at: Solid Mechanics Division, University of Waterloo, Ontario, Canada.

proved that the associated stability boundary cannot have convexity towards the region of stability. On the basis of this theorem one can readily arrive at some significant conclusions; thus the segment of a straight line joining any two points on the stability boundary, for instance, represents a safe stability estimate and a lower bound to the problem. The systems considered by Papkovich, however, exhibit a purely trivial fundamental solution involving no pre-buckling deflections.

The aim of this paper is to develop the concepts introduced in a previous work [1] and present a detailed discussion of the "general" case. In view of the very limited existing work concerned with general critical points, it is felt that the assessment of general buckling and post-buckling characteristics of such a system will provide useful information for many practical problems of this nature. Particular emphasis is given to the associated stability boundary and a concavity theorem for one-degree of freedom systems is proved. The theory is developed in terms of generalized coordinates, only elastic conservative systems being considered.

All the transformations used in the analysis are linear, and no attempt is made to obtain the results or part of the results in invariant form.

2. BASIC DEFINITIONS AND FORMULATION OF THE PROBLEM

The system under consideration is described by N generalized coordinates Q_i and M loading parameters Λ^j. Before proceeding with the development of the theory it will be convenient to introduce some of the geometrical concepts which will be used throughout this paper, and make some basic definitions which will help the understanding of the analytical approach.

Reference will often be made to two multi-dimensional spaces (manifolds) namely the M-dimensional *load–space* and $M+N$ dimensional *load–deflection space*, the former being a sub-space of the latter. It is assumed that the correspondence between the points of these spaces and all ordered sets of variables is unique.

For a given state of loading (i.e. for a given set of Λ^j) there will exist corresponding positions of equilibrium, and an equilibrium state can be associated with a point of $M+N$ dimensional load–deflection space. The entirety of these equilibrium points is called *the equilibrium surface*. It must be understood that, although a one-to-one correspondence between the points of the load–deflection space and a specific set of $M+N$ variables $(Q_i - \Lambda^j)$ is assumed to exist, it does not necessarily follow that there is also a one-to-one correspondence between a set of the loading parameters and the equilibrium states of the system. In fact, the points of the load–space will, in general, be associated with more than one equilibrium state which can be stable or unstable. Some of the points of the load–space may correspond to states of critical equilibrium and we shall call the entirety of these points *the critical surface*. Considering now a certain sequence of loading from the unloaded state, certain points of the critical surface will be associated with an initial loss of stability and we shall call the entirety of these points *the stability boundary*. In other words the stability boundary is a specific part of the critical surface. It will later be shown that no neighbouring equilibrium states exist beyond the stability boundary, and then the stability boundary can more aptly be termed *the existence boundary* demarcating *the regions of existence* and *inexistence*. It must be noted that the critical surface and consequently the stability boundary are defined in the loading space and not in the load-deflection space. In fact, they are the projections of *the critical zone* (consisting of critical equilibrium states) of the equilibrium surface into the loading space. This is illustrated on a strip of the equilibrium surface in Fig. 1. In general, the equilibrium surface can have a complicated shape, and at this stage we shall introduce the total potential energy of the system which will be the basis of the subsequent analysis and provide an analytical definition of the equilibrium surface. We assume that the total potential energy function

$$V = V(Q_i, \Lambda^j) \tag{1}$$

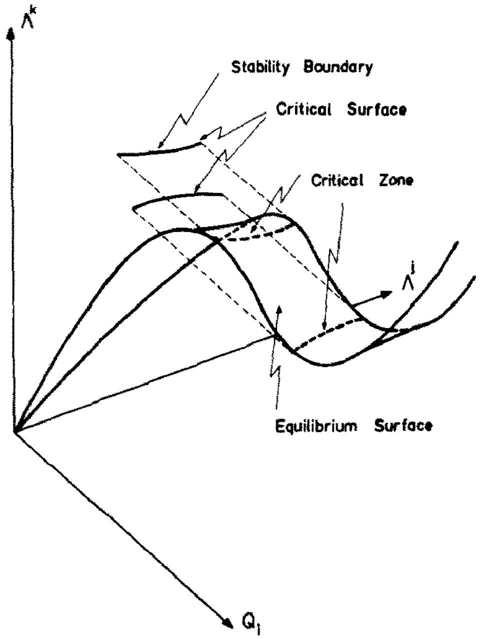

FIG. 1.

is single-valued and well-behaved at least in the region of interest. The N equilibrium equations

$$\frac{\partial V(Q_j, \Lambda^k)}{\partial Q_i} = 0 \tag{2}$$

define an M dimensional equilibrium surface in the $M+N$ dimensional load-deflection space. Suppose the equations (2) are solved simultaneously to yield results in the form

$$Q_i = Q_i(\Lambda^j), \tag{3}$$

and consider an arbitrarily chosen point F on the equilibrium surface representing a state of equilibrium $Q_i^F(\Lambda_F^j)$ which will be called fundamental. Using q_i and λ^j to denote increments in the variables Q_i and Λ^j respectively, the energy function can be referred to this fundamental state by writing it in the form

$$V = V(Q_i^F + q_i, \Lambda_F^j + \lambda^j). \tag{4}$$

We further introduce a linear, non-singular and orthogonal transformation,

$$q_i = \sum_{j=1}^{N} \alpha_{ij} u_j, \qquad |\alpha_{ij}| \neq 0, \tag{5}$$

to diagonalize the quadratic form of the energy function in the q_i. Introducing (5) into the equation (4) we get a new function

$$H(u_i, \lambda^j) \equiv V(Q_i^F + \alpha_{ij} u_j, \Lambda_F^j + \lambda^j). \tag{6}$$

New variables u_i are called "the principal generalized coordinates".

The energy function being diagonalized, the derivatives $H_{ii}(0,0) \equiv (\partial^2 H/\partial u_i^2)_0$ represent the Poincaré's stability coefficients and the "general" critical point is defined [1] as a point on the equilibrium surface where $H_{11} = 0$, $H_{ss} \neq 0$ for all $s \neq 1$ and $\text{grad}_\lambda H_1(0,0) \neq 0$.

Assuming that our fundamental state F coincides with a "general" critical point, we shall now introduce a transformation of the λ^i coordinates which will provide a canonical representation of the linear form corresponding to $(\partial^2 H/\partial u_1 \partial \lambda^i)|_0 \equiv H_1^i(0,0)$. Thus we choose a certain linear, non-singular and orthogonal transformation

$$\lambda^i = \sum_{j=1}^{M} \beta^{ij} \phi^j, \qquad |\beta^{ij}| \neq 0 \tag{7}$$

such that when it is substituted into the energy function $H(u_i, \lambda^j)$ the resulting transformed energy function

$$\pi(u_i, \phi^j) \equiv H(u_i, \textstyle\sum \beta^{ij} \phi^j) \tag{8}$$

will have the following properties:

$$\pi_1'(0,0) \neq 0, \pi_1^m(0,0) = 0 \tag{9}$$

for all $m \neq 1$. Here and elsewhere subscripts and superscripts denote partial differentiation with respect to the corresponding u_i and ϕ^j respectively, and all derivatives are evaluated at the fundamental state which is now a "general" critical point.

Clearly this transformation is based on the assumption that some coefficient $H_1^i(0,0)$ is not zero, and in fact this condition is satisfied by the definition of the general critical point (i.e. by $\text{grad}_\lambda H_1^i(0,0) \neq 0$). We also note that since the transformation (7) is related only to the λ^i, the quadratic form of the new function π in the u_i is still diagonalized. Thus

$$\pi_{ij}(0,0) = 0 \quad \text{for} \quad i \neq j \tag{10}$$

and the general critical point is defined simply by $\pi_{11} = 0$ and $\pi_{ss} \neq 0$ for $s \neq 1$.

3. EQUILIBRIUM SURFACE

We can now start the analysis with the new potential energy function $\pi(u_i, \phi^j)$, the only necessary properties of which are given by the equations (9) and (10). The solution of N equilibrium equations $\pi_i(u_j, \phi^k)$ can be described in parametric form by the functions,

$$\begin{aligned} u_i &= u_i(\varepsilon^k) \\ \phi^j &= \phi^j(\varepsilon^k), \end{aligned} \tag{11}$$

of M unspecified parameters ε^k which must be chosen so that the functions (11) are single-valued. The parameters ε^k can subsequently be equated to appropriate variables according to the nature of the problem under consideration [2]. In order to avoid further complication and due to the fact that this paper is devoted to the study of a general critical point only, we shall choose the independent variables in the beginning, this choice being guided by the authors previous work [1]. (See also Thompson's paper [12].) Thus we choose the critical coordinate u_1 and transformed loading parameters $\phi^m (m \neq 1)$ as M independent variables and have the functions in the form

$$u_s = u_s(u_1, \phi^m), \qquad \phi' = \phi'(u_1, \phi^m) \tag{12}$$

where $s \neq 1$, $m \neq 1$. Substituting these functions back into the equilibrium equations $\pi_i = 0$ we get

$$\pi_i[u_j(u_1, \phi^m), \qquad \phi^k(u_1, \phi^m)] = 0 \tag{13}$$

in which although we write $u_i(u_1, \phi^m)$ and $\phi^k(u_1, \phi^m)$ for the sake of tidiness, in fact, only u_s and ϕ' are the functions of u_1 and ϕ^m while all the $\phi^m (m \neq 1)$ and u_1 are independent variables. It is further understood that the left-hand sides of (13) are the identically zero functions of the independent variables and can therefore be differentiated with respect to u_1 and ϕ^m as many times as we please.

Thus differentiating (13) once with respect to u_1 and once with respect to $\phi^m (m = 2, 3, \cdots, M)$ we get

$$\left.\begin{aligned} \pi_{ij}u_{j,1} + \pi_i^j \phi_1^j &= 0, \\ \pi_{ij}u_j^m + \pi_i^j \phi^{j,m} &= 0 \end{aligned}\right\} \tag{14}$$

where differentiation of variables are indicated by superscripts and subscripts with commas, and summation convention is adopted both for subscripts and superscripts separately so that summations over subscripts range from 1 to N and over superscripts from 1 to M.

Evaluating (14) at the critical point F we have

$$\left.\begin{aligned} \phi'_1 &= 0 \\ \phi'^{,m} &= 0 \end{aligned}\right\} \text{ for } i = 1 \tag{15}$$

and

$$\left.\begin{aligned} u_{s,1} &= 0 \\ u_s^m &= -\frac{\pi_s^m}{\pi_{ss}} \end{aligned}\right\} \text{ for } i = s. \tag{16}$$

Differentiating the first of the equations (14) with respect to u_1 we have

$$(\pi_{ijk}u_{k,1} + \pi_{ij}^k \phi_1^k)u_{j,1} + \pi_{ij}u_{j,11} + (\pi_{ik}^j u_{k,1} + \pi_i^{jk}\phi_1^k)\phi_1^j + \pi_i^j \phi_{11}^j = 0 \tag{17}$$

which when evaluated at the point F yields for $i = 1$

$$\phi'_{11} = -\frac{\pi_{111}}{\pi'_1}$$

and for $i = s$

$$u_{s,11} = -\frac{1}{\pi_{ss}}\left(\pi_{s11} - \frac{\pi'_s}{\pi'_1}\pi_{111}\right). \tag{18}$$

Differentiating again the first of equations (14) this time with respect to $\phi^m(m = 2, 3, \cdots, M)$ we get

$$(\pi_{ijk}u_k^m + \pi_{ij}^k\phi^{k,m})u_{j,1} + \pi_{ij}u_{j,1}^m + (\pi_{ik}^j u_k^m + \pi_i^{jk}\phi^{k,m})\phi_1^j + \pi_i^j\phi_1^{j,m} = 0 \tag{19}$$

which upon evaluation yields for $i = 1$

$$\phi_1^{\prime,m} = -\frac{1}{\pi'_1}\left(\pi_{11}^m - \pi_{11s}\frac{\pi_s^m}{\pi_{ss}}\right). \tag{20}$$

Now we differentiate the second of the equations (14) with respect to $\phi^n(n = 2, 3, \cdots, M)$ to get

$$(\pi_{ijk}u_k^n + \pi_{ij}^k\phi^{k,n})u_j^m + \pi_{ij}u_j^{mn} + (\pi_{ik}^j u_k^n + \pi_i^{jk}\phi^{k,n})\phi^{j,m} + \pi_i^j\phi^{j,mn} = 0. \tag{21}$$

Evaluating at the critical point F and using the equations (15) and (16) we get for $i = 1$

$$\phi^{\prime,mn} = -\frac{1}{\pi'_1}\left(\pi_1^{mn} + \pi_{1sr}\frac{\pi_s^m\pi_r^n}{\pi_{ss}\pi_{rr}} - \frac{\pi_{1s}^n\pi_s^m + \pi_{1s}^m\pi_s^n}{\pi_{ss}}\right) \tag{22}$$

where $s \neq 1, r \neq 1, m \neq 1, n \neq 1$.

We are now in a position to derive the asymptotic relationship $\phi' = \phi'(u_1, \phi^m)$ and $u_s = u_s(u_1, \phi^m)$. Thus, using Taylor's expansion and the derivatives (15), (16), (18), (20) and (22) we have

$$\pi'_1\phi' + \frac{1}{2}\pi_{111}u_1^2 + \left(\pi_{11}^m - \pi_{11s}\frac{\pi_s^m}{\pi_{ss}}\right)u_1\phi^m + \frac{1}{2}\left(\pi_1^{mn} + \pi_{1sr}\frac{\pi_s^m\pi_r^n}{\pi_{ss}\pi_{rr}} - 2\frac{\pi_{1s}^m\pi_s^n}{\pi_{ss}}\right)\phi^m\phi^n = 0 \tag{23}$$

and

$$\pi_{ss}u_s + \pi_s^m\phi^m + \frac{1}{2}\left(\pi_{s11} - \frac{\pi'_s}{\pi'_1}\pi_{111}\right)u_1^2 = 0 \tag{24}$$

where summation convention, as described before, is adopted. Using (23), the equation (24) can also be written in the form

$$\pi_{ss}u_s + \pi_s^m\phi^m + \left(\pi'_s - \frac{\pi_{s11}\pi'_1}{\pi_{111}}\right)\phi' = 0. \tag{25}$$

The equations (23) and (25), when considered together, define the M dimensional equilibrium surface in the vicinity of the fundamental state F, and when considered individually they represent the projections of this surface into the $u_1 - \phi^i$ and $u_s - \phi^i$ subspaces respectively.

Suppose we take a general ray given by $\phi^i = l^i\xi$ where at least $l' \neq 0$. The equilibrium equations (23) and (25) will then yield

$$\pi'_1 l'\xi + \tfrac{1}{2}\pi_{111}u_1^2 = 0 \tag{26}$$

and

$$\pi_{ss}u_s + \left[\pi_s^m l^m + \left(\pi_s' - \frac{\pi_{s11}\pi_1'}{\pi_{111}}\right) l'\right]\xi = 0 \tag{27}$$

which indicate a limit point on a plot of u_1 against ξ.

On the other hand, suppose we take a special ray defined by $\phi^i = l^i\xi$ where $l' = 0$ and $l^m \neq 0$; the equations (23) and (25) now yield

$$u_1 = \frac{1}{\pi_{111}}[-a \pm (a^2 - \pi_{111}b)^{\frac{1}{2}}]\xi \tag{28}$$

and

$$u_s = -\frac{\pi_s^m}{\pi_{ss}} l^m \xi \tag{29}$$

in which a and b are constants and given by

$$\begin{aligned} a &= \left(\pi_{11}^m - \pi_{s11}\frac{\pi_s^m}{\pi_{ss}}\right) l^m \equiv c^m l^m \\ b &= \left(\pi_1^{mn} + \pi_{1sr}\frac{\pi_s^m \pi_r^n}{\pi_{ss}\pi_{rr}} - 2\frac{\pi_{1s}^m \pi_s^n}{\pi_{ss}}\right) l^m l^n \equiv \mathrm{d}^{mn} l^m l^n. \end{aligned} \tag{30}$$

(28) obviously indicates bifurcation on a plot of ξ against u_1 provided

$$a^2 - \pi_{111}b > 0. \tag{31}$$

It is thus observed that at a general critical point both an extremum and bifurcation can be obtained on different plots, this depending on the shape of the equilibrium surface. In order to visualize this surface we shall consider the special case of two loads, namely ϕ' and ϕ^2; then (23) becomes the equation of a

Synclastic	< 0	(32a)
Anticlastic surface if $a^2 - \pi_{111}b$	> 0	(32b)
Parabolic	$= 0.$	(32c)

These surfaces are illustrated in Figs. 2(a), (b) and (c) respectively. Using a similar terminology the critical point F can be called *elliptic* if (32a) holds, *hyperbolic* if (32b) holds and *parabolic* if (32c) holds. It is clear that in the event of elliptic and parabolic points bifurcation is ruled out while a hyperbolic point can be regarded a limit point on a plot of $u_1 - \phi'$, and a bifurcation point on a plot $u_1 - \phi^2$.

Extending the terminology used for two-dimensional surfaces to multi-dimensional surfaces, we shall call the surface (23) *synclastic* if (32a) holds for all possible rays given by $\phi^i = l^i\xi$ where $l' = 0$, some $l^m \neq 0\,(m \neq 1)$ in which case the matrix $[c^m c^n - \pi_{111}d^{mn}]$ is negative definite. The surface (23) will be called *anticlastic* if (32b) holds for all rays defined above in which case the matrix $[c^m c^n - \pi_{111}d^{mn}]$ is positive definite. Similarly the surface (23) will be called *parabolic* if (32c) holds for all rays defined above in which case the matrix $[c^m c^n - \pi_{111}d^{mn}]$ becomes a null matrix. Clearly, if the surface is not synclastic, it does not necessarily follow that the surface is either anticlastic or parabolic. In fact, when the matrix $[c^m c^n - \pi_{111}d^{mn}]$ is indefinite, the surface takes a quite arbitrary shape. One of the

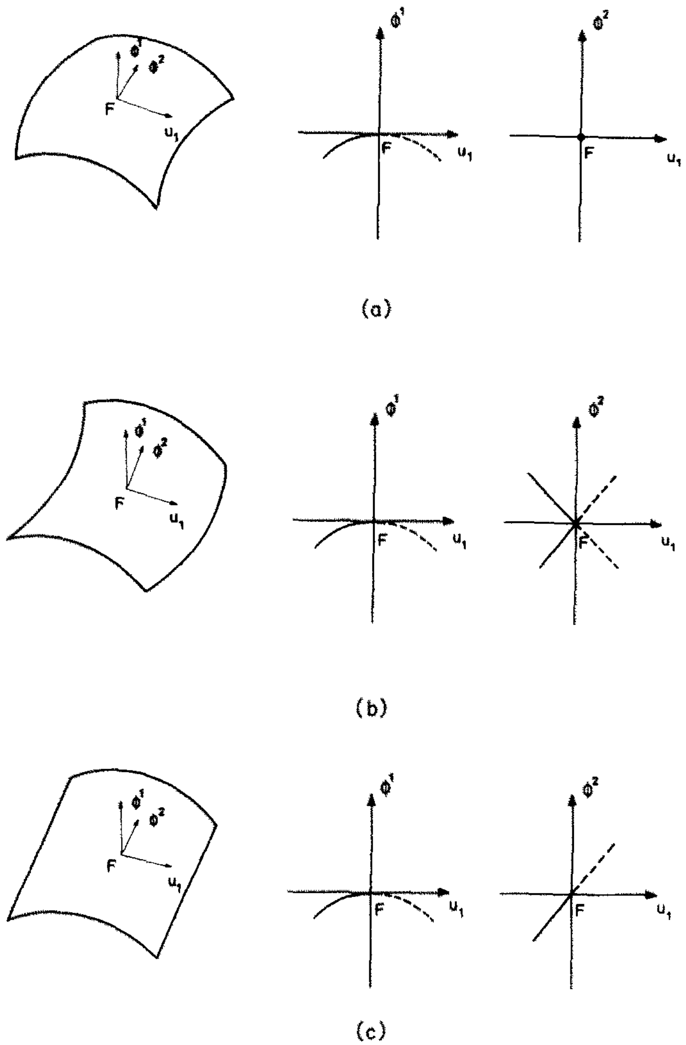

FIG. 2.

reasons we introduce the above terminology is to ease the reference to these particular shapes which will often arise.

4. CRITICAL ZONE AND STABILITY BOUNDARY

In the preceding section the shape of the equilibrium surface around a general critical point has been investigated. In this section we proceed to determine other critical points which might exist in the vicinity of the point F on the equilibrium surface.

Critical points should satisfy the determinantal equation

$$\Delta = |\pi_{ij}(u_k, \phi^l)| \equiv |H_{ij}(u_k, \lambda^l)| \equiv |\alpha_{ij}|^2 |V_{ij}(q_k, \lambda^l)| = 0. \tag{33}$$

Differentiating the determinant $\Delta(u_k, \phi^l) = |\pi_{ij}(u_k, \phi^l)|$ by columns (or by rows) once with respect to u_i and once with respect to ϕ^j and evaluating at the point F we get

$$\left.\begin{aligned} \Delta_i &= \pi_{i11}\prod_{s=2}^{N}\pi_{ss} \\ \Delta^j &= \pi^j_{11}\prod_{s=2}^{N}\pi_{ss} \end{aligned}\right\} \tag{34}$$

which will be used later in the analysis.

Considering the equation (33) together with the equilibrium equations, we see that the solution of these $N+1$ equations can be described by $N+1$ functions of $M-1$ independent parameters which we choose as $\phi^m (m \neq 1)$. Thus the functions take the form

$$\overset{*}{u}_i = \overset{*}{u}_i(\overset{*}{\phi}{}^m), \qquad \overset{*}{\phi}{}' = \overset{*}{\phi}{}'(\overset{*}{\phi}{}^m) \tag{35}$$

where a star is used to denote the critical variables. If these functions are substituted into the equilibrium equations $\pi_i = 0$ and into the equation (33) we obtain the conditions of critical equilibrium as

$$\left.\begin{aligned} \pi_i[\overset{*}{u}_j(\overset{*}{\phi}{}^m), \overset{*}{\phi}{}'(\overset{*}{\phi}{}^m), \overset{*}{\phi}{}^m] &= 0, \\ \Delta[\overset{*}{u}_i(\overset{*}{\phi}{}^m), \overset{*}{\phi}{}'(\overset{*}{\phi}{}^m), \overset{*}{\phi}{}^m] &= 0. \end{aligned}\right\} \tag{36}$$

Differentiating these functions with respect to $\overset{*}{\phi}{}^m (m = 2, \cdots, M)$ we have

$$\left.\begin{aligned} \pi_{ij}\overset{*}{u}{}^m_j + \pi'_i\overset{*}{\phi}{}'^{,m} + \pi^m_i &= 0 \\ \Delta_i\overset{*}{u}{}^m_i + \Delta'\overset{*}{\phi}{}'^{,m} + \Delta^m &= 0 \end{aligned}\right\} \tag{37}$$

which on evaluating at the critical point F yield

$$\left.\begin{aligned} \overset{*}{\phi}{}'^{,m} &= 0 \\ \overset{*}{u}{}^m_s &= 0 \\ \overset{*}{u}{}^m_1 &= -\frac{1}{\pi_{111}}\left(\pi^m_{11} - \pi_{s11}\frac{\pi^m_s}{\pi_{ss}}\right). \end{aligned}\right\} \tag{38}$$

It is noted that subscripts and superscripts on the variables now denote differentiation with respect to the corresponding generalized coordinate and critical loading parameter respectively.

Differentiating the first of the equations (36) for a second time with respect to $\overset{*}{\phi}{}^n (n = 2, 3, \cdots, M)$ we get

$$\begin{aligned} &(\pi_{ijk}\overset{*}{u}{}^n_k + \pi'_{ij}\overset{*}{\phi}{}'^{,n} + \pi^n_{ij})\overset{*}{u}{}^m_j + \pi_{ij}\overset{*}{u}{}^{mn}_j + (\pi'_{ij}\overset{*}{u}{}^n_j + \pi''_i\overset{*}{\phi}{}'^{,n} + \pi^n_i)\overset{*}{\phi}{}'^{,m} + \pi'_i\overset{*}{\phi}{}'^{,mn} \\ &\quad + \pi^m_{ij}\overset{*}{u}{}^n_j + \pi^{m\prime}_i\overset{*}{\phi}{}'^{,n} + \pi^{mn}_i = 0. \end{aligned} \tag{39}$$

Evaluating at the point F, (39) yields

$$\begin{aligned} &\pi_{111}\overset{*}{u}{}^m_1\overset{*}{u}{}^n_1 + \pi_{11s}\overset{*}{u}{}^n_1\overset{*}{u}{}^m_s + \pi_{11s}\overset{*}{u}{}^n_s\overset{*}{u}{}^m_1 + \pi_{1sr}\overset{*}{u}{}^n_s\overset{*}{u}{}^n_r \\ &\quad + \pi^n_{11}\overset{*}{u}{}^m_1 + \pi^n_{1s}\overset{*}{u}{}^m_s + \pi'_1\overset{*}{\phi}{}'^{,mn} + \pi^m_{11}\overset{*}{u}{}^n_1 + \pi^m_{1s}\overset{*}{u}{}^n_s + \pi^{mn}_1 = 0. \end{aligned} \tag{40}$$

Substituting for $\overset{*}{u}_1^m$ and $\overset{*}{u}_s^m$ in (40) we get

$$\overset{*}{\phi}{}^{\prime,mn} = \frac{1}{\pi'_1\pi_{111}}(c^m c^n - \pi_{111}d^{mn}) \tag{41}$$

where c^m and d^{mn} are constants and defined by (30).

The asymptotic relationships can now be written in the form

$$\overset{*}{\phi}{}' = \frac{1}{2\pi'_1\pi_{111}}(c^m c^n - \pi_{111}d^{mn})\overset{*}{\phi}{}^m\overset{*}{\phi}{}^n, \tag{42}$$

$$\overset{*}{u}_1 = -\frac{1}{\pi_{111}}\left(\pi^m_{11} - \pi_{s11}\frac{\pi^m_s}{\pi_{ss}}\right)\overset{*}{\phi}{}^m \equiv -\frac{1}{\pi_{111}}c^m\overset{*}{\phi}{}^m \tag{43}$$

and

$$\overset{*}{u}_s = -\frac{\pi^m_s}{\pi_{ss}}\overset{*}{\phi}{}^m \tag{44}$$

which, when considered together, define the *critical zone* of the equilibrium surface in the vicinity of the critical point. When considered individually, (42) represents the *stability boundary* (in general the *critical surfaces*) while the equations (43) and (44) represent the projections of the critical zone in the $u_1 - \phi^m$ and $u_s - \phi^m$ subspaces respectively.

Now suppose the fundamental state is an elliptic point satisfying (32a) for all rays given by $\phi^m = l^m\xi$ $(m \neq 1)$, then, the matrix $[c^m c^n - \pi_{111}d^{mn}]$ is negative definite and hence the stability boundary (42) is a synclastic surface concave towards the region of existence which is identified by the curvature $\phi'_{11}|_F = -\pi_{111}/\pi'_1$ obtained from the equilibrium equation (23) on a plot of u_1 against ϕ'. In the majority of practical problems it becomes concave towards the origin of the loading space.

On the other hand if the fundamental state is a hyperbolic point the stability boundary is again synclastic but convex towards the region of existence. If F is a parabolic point the stability boundary becomes a plane.

It must be noted that in general the stability boundary might not be synclastic, and this happens when the matrix $[c^m c^n - \pi_{111}d^{mn}]$ is indefinite, this possibility being obviously ruled out in case of two loading parameters (ϕ' and ϕ^2) only.

We clearly see, however, that if the matrix $[c^m c^n - \pi_{111}d^{mn}]$ is positive or positive definite i.e. if

$$a^2 - \pi_{111}b \leq 0 \tag{45}$$

the stability boundary cannot have convexity towards the region of existence. Since the condition (45) also eliminates (see section 3) the possibility of bifurcation, a Theorem has been established:

THEOREM 1

If the critical point is a genuine limit point at which bifurcation is ruled out, then the associated stability boundary cannot have convexity towards the region of existence.

COROLLARY 1

A necessary condition that the stability boundary is convex with regard to the region of existence is that the possibility of bifurcation at the associated critical point must not be ruled out.

COROLLARY 2

In the particular case of two loads (ϕ' and ϕ^2) only, the aforementioned condition (corollary 1) is both necessary and sufficient. i.e. If bifurcation at the critical point is possible then the stability boundary is convex with respect to the region of existence.

5. STABILITY BOUNDARY OF A ONE-DEGREE OF FREEDOM SYSTEM

So far, the potential energy function V and consequently the functions H and π were assumed to be general in the sense that they were not linear in the loading parameters. These parameters can, therefore, represent some other variables such as imperfections. In the majority of structural problems, however, the potential energy V is linear in external loads, and if the λ^i are restricted to represent the external loading only, we can conclude on the basis of the transformation (7) that all the second and higher order derivatives of the function π with respect to the ϕ^i should vanish. Thus

$$\pi_i^{jk} = \pi_i^{jkl} = \cdots = \pi_{ij}^{kl} = \pi_{ij}^{klm} = \cdots = 0. \tag{46}$$

Using (46) and assuming that we have only two loads, namely the ϕ' and ϕ^2, (the proof for the case of M loads will be given elsewhere) we obtain the stability boundary of a one-degree of freedom system as

$$\overset{*}{\phi}{}' = \frac{(\pi_{11}^2)^2}{2\pi'_1\pi_{111}}(\overset{*}{\phi}{}^2)^2 \tag{47}$$

and the equilibrium surface as

$$\pi'_1\phi' + \tfrac{1}{2}\pi_{111}u_1^2 + \pi_{11}^2 u_1\phi^2 = 0 \tag{48}$$

where brackets are used to denote the square of variables with superscripts.

On a comparison of the curvatures

$$\overset{*}{\phi}{}'^{,22}|_F = \frac{(\pi_{11}^2)^2}{\pi'_1\pi_{111}} \text{ and } \phi'_{11}|_F = -\frac{\pi_{111}}{\pi'_1}$$

we see that the stability boundary is convex towards the region of existence. Hence the following theorem for a one-degree of freedom system with two loads is proved;

THEOREM 2

The stability boundary of a one-degree of freedom system with two loads, cannot have convexity towards the region of inexistence.

6. CRITICAL SURFACES AS THE EXISTENCE BOUNDARY

It is noted that we are using the curvature $\phi'_{11}|_F = -\pi_{11}/\pi'_1$ as a criterion to determine the regions of existence and inexistence, but this point needs further clarification.

Evidently, if $\phi^m = 0\,(m \neq 1)$ the real equilibrium states in the vicinity of F can only exist for either $\phi' > 0$ or $\phi' < 0$ depending on the sign of the coefficients π_{111} and π'_1. We shall now show that the critical surfaces (and in particular the stability boundary) constitute an existence boundary in the sense that real equilibrium states can only exist for the points (of the load–space) lying on one side of the critical surface, there being no neighbouring equilibrium states corresponding to the points lying on the other side of the critical surface.

Considering an arbitrarily chosen point A on the critical surface defined by a set of critical parameters $\overset{*}{\phi}{}^i_A$ $(i = 1, 2, \cdots, M)$ we shall examine the corresponding equilibrium states by keeping $\phi^m = \overset{*}{\phi}{}^m_A = \text{const}\ (m \neq 1)$ and giving a small but finite increment ε to $\overset{*}{\phi}{}'_A$. For certain values of ε, a set of parameters

$$\phi^m = \overset{*}{\phi}{}^m_A, \quad \phi' = \overset{*}{\phi}{}'_A + \varepsilon$$

defines a point in the original load–space (Λ^i space).

Substituting for ϕ' and ϕ^m in the equilibrium equation (23) we get

$$\pi'_1 \overset{*}{\phi}{}'_A + \pi'_1 \varepsilon + \tfrac{1}{2}\pi_{111} u_1^2 + c^m u_1 \overset{*}{\phi}{}^m_A + \tfrac{1}{2} d^{mn} \overset{*}{\phi}{}^m_A \overset{*}{\phi}{}^n_A = 0. \tag{49}$$

Using (42) and (43) which must be satisfied by all the critical parameters we have

$$u_1 = \overset{*}{u}{}^A_1 \pm \frac{1}{\pi_{111}}(-2\pi_{111}\pi'_1\varepsilon)^{\frac{1}{2}}. \tag{50}$$

Now suppose π_{111} and π'_1 have the same sign, then, real equilibrium states can only exist for $\varepsilon < 0$. As the point A slides on the critical surface, $\varepsilon < 0$ will define the *region of existence* and $\varepsilon > 0$ (the other side of the critical surface) will define the *region of inexistence.* On the other hand if π_{111} and π'_1 have the opposite sign, real solutions can only exist for $\varepsilon > 0$ defining the region of existence. Obviously $\varepsilon = 0$ gives the point A.

Thus it can be concluded that the critical surface is, in fact, an existence boundary and regions of existence and inexistence can be located by examining the sign of the curvature $\phi'_{11}|_F$. This is a general result which is valid regardless of the shape of the equilibrium surface. On the basis of afore going theory the following theorem can be stated:

THEOREM 3

The critical surfaces constitute an existence boundary so that neighbouring equilibrium states can only exist for the points (of the load–space) lying on one side of the critical surfaces.

7. STABILITY OF EQUILIBRIUM

In order to examine the stability of the equilibrium states in the vicinity of F we suppose that F is *general primary* so that $\pi_{11} = 0$, $\pi_{ss} > 0\ (s \neq 1)$.

For the stability of an equilibrium state, it is required that the potential energy function have a complete relative minimum at that point. If the state is non-critical the necessary and sufficient condition of stability is the positive definiteness of the second variation of energy. This implies that the stability determinant

$$\Delta(u_k, \phi^l) = |\pi_{ij}(u_k, \phi^l)|$$

evaluated at that point must be positive. If the stability determinant is negative, then, the equilibrium state is unstable. It must be noted, however, that the assessment of stability by examining the sign of the stability determinant is only valid in the vicinity of a *primary* critical point since the stability coefficients cannot change sign in this small region before passing through zero.

Expanding $\Delta(u_k, \phi^l)$ into Taylor series around F, and using (34) we have

$$\Delta = (\pi_{11i}u_i + \pi^i_{11}\phi^i) \prod_{s=2}^{N} \pi_{ss} + \tfrac{1}{2}[\cdots] + \cdots \tag{51}$$

Evaluating (51) at an arbitrary equilibrium point which is defined by the equations (23) and (24), and keeping to a first approximation we get

$$\Delta = (\pi_{111} u_1 + c^m \phi^m) \prod_{s=2}^{N} \pi_{ss}. \tag{52}$$

Since we are restricting attention to the neighbourhood of a primary critical point where $\pi_{ss} > 0\,(s \neq 1)$ we have the following stability criterion

$$\pi_{111} u_1 + c^m \phi^m \begin{cases} > & \text{stable} \\ = 0 & \text{for critical equilibrium.} \\ < & \text{unstable} \end{cases} \tag{53}$$

Consider now a critical point A on the critical zone satisfying the equations (42), (43), and (44). Neighbouring equilibrium states can be obtained by keeping $\overset{*}{\phi}{}^m_A\,(m \neq 1)$ constant and giving a small but finite increment ε to $\overset{*}{\phi}{}'_A$. Thus for certain values of ε, the set of

$$\left.\begin{aligned} \phi^m &= \overset{*}{\phi}{}^m_A \\ \phi' &= \overset{*}{\phi}{}'_A + \varepsilon \end{aligned}\right\} \tag{54}$$

corresponds to certain equilibrium states. u_1 was determined in the preceding section [see the equations (50)]. Using (25), (44) and (54) u_s can also be determined as

$$u_s = \overset{*}{u}{}^A_s - \frac{1}{\pi_{ss}}\left(\pi'_s - \frac{\pi_{s11}\pi'_s}{\pi_{111}}\right)\varepsilon. \tag{55}$$

Here we are only interested in real equilibrium states, and it will therefore be assumed that the increment ε is given in the right direction to ensure real solutions.

Evaluating the stability determinant at the particular states defined by (50), (54) and (55) we get

$$\Delta = [\pi_{111}\overset{*}{u}{}^A_1 \pm (2\pi_{111}\pi'_1\varepsilon)^{\frac{1}{2}} + c^m \overset{*}{\phi}{}^m_A] \prod_{s=2}^{N} \pi_{ss} \tag{56}$$

and using (43) we have

$$\Delta = \pm(-2\pi_{111}\pi'_1\varepsilon)^{\frac{1}{2}} \prod_{s=2}^{N} \pi_{ss}. \tag{57}$$

Evidently (57) is positive for one solution and negative for the other lying on the opposite side of the critical zone. Moving the point A along the critical zone and assigning certain values to ε, the stability of all equilibrium states in the vicinity of F can be examined. The obvious conclusion can be stated as a

THEOREM 4

The equilibrium surface is divided by the critical zone into two domains so that on one side of the critical zone the equilibrium is stable and on the other side the equilibrium is unstable.

8. DISCUSSION AND CONCLUSIONS

A detailed non-linear stability analysis of discrete conservative structural systems, whose loss of stability is associated with a "general" critical point (see Ref. [1]), is presented. The results demonstrate the general buckling and post-buckling characteristics of the system, and are particularly concerned with the associated stability boundary:

1. It is shown that (Theorem 1), if a general critical point is a genuine limit point at which bifurcation of solution is ruled out, then, the stability boundary cannot have convexity towards *the region of existence.* Two corollaries follow immediately:

(a). A necessary condition that the stability boundary is convex towards the region of existence is that bifurcation of solution at the general critical point must be possible.

(b). In the particular case of two loads only, this condition is both necessary and sufficient.

The investigation of Loo and Evan-Ivanowski on the buckling of a shallow spherical shell under the action of combined pressure and concentrated load at the apex provides an example for this theorem. The authors conclude that the loss of stability of such a system is always associated with an *elliptic* general critical point at which bifurcation is ruled out, and hence the behaviour of the system must be in compliance with Theorem 1. In fact the stability boundary reported is concave towards the region of existence as expected.

2. It is observed that a one-degree of freedom system exhibits certain additional buckling characteristics allowing an even stronger theorem to be proved:

The stability boundary of a one-degree of freedom system with two loads cannot have convexity towards the region of inexistence.

The work of Fung and Kaplan [13] on the buckling of a sinusoidal arch under a sinusoidal load and having an initial thrust can be mentioned as a particular example for this theorem. The buckling behaviour of this arch, when its geometrical properties allow the "general" critical points to take place, is analogous to that of a one-degree of freedom system; and the associated stability boundary is, in fact, convex towards the region of existence.

Although the preceeding two results are asymptotic, thus being valid in the vicinity of a fundamental state, it is intuitive that if a surface is convex at every point it is also convex in its entirety. The theorems will, therefore, hold for the entire stability boundary provided the loss of stability is always associated with "general" critical points under various combinations of loads. This conclusion has some practical consequences; thus, an upper bound (lower bound) to the stability boundary can readily be obtained on the basis of Theorem 2 (Theorem 1) if any two points on it are known (by simply joining these two points).

3. Another interesting general result is Theorem 3:

The critical surfaces (and in particular the stability boundary) constitute an *existence boundary*, so that neighbouring equilibrium states can only exist for the points (of the loading space) lying on one side of the critical surfaces.

One of the immediate consequences of this theorem is that the loss of stability of the system will normally be associated with snap-through type buckling.

4. It is also shown that the *critical zone* divides the equilibrium surface into two domains so that on one side of the critical zone the equilibrium is stable and on the other side the equilibrium is unstable. In the proof presented for this stability distribution on the equilibrium surface, the critical zone consists of *general primary*

critical points which are associated with an initial loss of stability. Thus an initially stable surface becomes unstable beyond the critical zone.

Finally, it is expected that the general buckling and post-buckling characteristics discussed in this paper will provide an insight into many particular problems of this nature, and the theorems established (see also Ref. [14]) will prove useful particularly in those situations in which the determination of the stability boundary poses serious fundamental difficulties and a lower or upper bound is sought.

Acknowledgements—The author is grateful to Dr. J. M. T. Thompson, University College, London for many valuable discussions, and to The Scientific and Technical Research Council of Turkey for their financial support of the work.

REFERENCES

[1] K. Huseyin, Fundamental principles in the buckling of structures under combined loading. *Int. J. Solids Struct.* **5**, 479 (1969).
[2] K. Huseyin, Elastic stability of structures under combined loading, Ph.D. Thesis, University of London (1968).
[3] R. M. Evan-Ivanowski and T. C. Loo, Deformations and stability of spherical shells under the action of combined concentrated loads and uniform pressure. Tech. Rep., S.U.R.I.834(II), No. 4 (1962).
[4] T. C. Loo and R. M. Evan-Ivanowski, Interaction of critical pressures and critical concentrated loads acting on shallow spherical shells. *J. appl. Mech.* **33**, (1966).
[5] A. B. Sabir, Large deflection and buckling behaviour of a spherical shell with inward point load and uniform pressure. *J. Mech. engng Sci.* **6**, 394 (1964).
[6] W. T. Koiter, Over de Stabiliteit van het Elastische Evenwicht. Thesis, Delft (1945).
[7] J. M. T. Thompson, Basic principles in the general theory of elastic stability. *J. Mech. Phys. Solids* **11**, 13 (1963).
[8] J. Roorda, The buckling behaviour of imperfect structural systems. *J. Mech. Phys. Solids* **13**, 267 (1965).
[9] A. H. Chilver, Coupled modes of buckling. *J. Mech. Phys. Solids* **15**, 15 (1967).
[10] M. J. Sewell, On the connexion between stability and the shape of the equilibrium surface. *J. Mech. Phys. Solids* **14**, 203 (1966).
[11] P. F. Papkovich, Works on the structural mechanics of ships. 4 (1963).
[12] J. M. T. Thompson, Discrete branching points in the general theory of elastic stability. *J. Mech. Phys. Solids* **13**, 295 (1965).
[13] Y. C. Fung and A. Kaplan, Buckling of low arches or curved beams of small curvature. N.A.C.A.,T.N. 2840 (1952).
[14] K. Huseyin, The convexity of the stability boundary of symmetric structural systems. *Acta Mech.* **8**, in press.

(*Received* 17 *February* 1969; *revised* 2 *May* 1969)

Абстракт—Исследуется устойчивость общей упругой строительной системы, описанной и обобщенными координатами и M параметрами нагрузки. Сосредоточивается в области кенсорвативных систем, которых потера устойчивости связана с "общей" критической точкой. Предлагаются существенно определенные ассимптотические зависимости, описывающие поверхность равновесия в соседстве критической точки. Введение некоторых преобразований позваляет удобно исследовать локальные свойства поверхности равновесия и природу потери устойчивости. Проверятся теорема, касающаяся распределения устойчивости на поверхности равновесия.

Обращается специальное внимание на предел устойчивости системы. Оказывается, что в данном случае существует строгая зависимость между типом критической точки и формой предела устойчивости. Предлагаются три теоремы, касающиеся основных свойств предела устойчивости.